1. Bizonyítsd be, hogy

a) $1^2 - 2^2 + 3^2 - ... + (-1)^{n-1}n^2 = (-1)^{n-1}\frac{n(n+1)}{2}$

b) $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$

c) $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$

d) $1^2 + 3^2 + ... + (2n-1)^2 = \frac{n(4n^2-1)}{3}$

e) $1^3 + 2^3 + 3^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$

f) $1^4 + 2^4 + 3^4 + ... + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

g) $1 + x + x^2 + ... + x^n = \frac{x^{n+1}-1}{x-1}, \quad x \neq 1$

h) $\frac{1^4}{1\cdot 3} + \frac{2^4}{3\cdot 5} + ... + \frac{n^4}{(2n-1)(2n+1)} = \frac{n(n+1)(n^2+n+1)}{6(2n+1)}$

i) $1\cdot 2 + 2\cdot 5 + ... + n\cdot(3n-1) = n^2\cdot(n+1)$

j) $\frac{1}{1\cdot 3\cdot 5} + \frac{2}{3\cdot 5\cdot 7} + ... + \frac{n}{(2n-1)(2n+1)(2n+3)} = \frac{n(n+1)}{2(2n+1)(2n+3)}$

k) $1\cdot 2\cdot 3 ...\cdot k + 2\cdot 3 ...\cdot k\cdot(k+1) + ... + n\cdot(n+1)...\cdot(n+k-1) = \frac{n(n+1)(n+2)...(n+k)}{k+1}, \quad k \geq 2$

l) $log_x\sqrt{a} + log_{x^2}\sqrt[3]{a} + log_{x^3}\sqrt[4]{a} + ... + log_{x^{n-1}}\sqrt[n]{a} = log_x\sqrt[n+1]{a^n}$

2. Mivel egyenlő a következő sorok összege? Bizonyítsd be ez egyenlőséget!

a) $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + ... + n\cdot(n+1) =$

b) $1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + ... + n\cdot(n+1)\cdot(n+2) =$

c) $\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + ... + \frac{1}{n\cdot(n+1)\cdot(n+2)} =$

d) $\frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{2\cdot 3\cdot 4\cdot 5} + ... + \frac{1}{n\cdot(n+1)\cdot(n+2)\cdot(n+3)} =$

e) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + ... + \frac{1}{n\cdot(n+1)} =$

f) $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + ... + \frac{1}{(2n-1)\cdot(2n+1)} =$

g) $\frac{1^2}{1\cdot 3} + \frac{2^2}{3\cdot 5} + ... + \frac{n^2}{(2n-1)\cdot(2n+1)} =$

h) $\frac{1}{1\cdot 4} + \frac{1}{4\cdot 7} + ... + \frac{1}{(3n-2)\cdot(3n+1)} =$

i) $\frac{1}{1\cdot 5} + \frac{1}{5\cdot 9} + ... + \frac{1}{(4n-3)\cdot(4n+1)} =$

j) $\frac{1}{k\cdot(k+1)} + \frac{1}{(k+1)\cdot(k+2)} + ... + \frac{1}{(k+n-1)\cdot(k+n)} =$

k) $\sum_{k=1}^{n} k\cdot k!$

l) $\sum_{k=1}^{n} \frac{k!}{k^2+k+1}$

m) $\sum_{k=1}^{n} \frac{k}{(k+1)!}$

n) $\sum_{k=1}^{n} \frac{k+2}{k!+(k+1)!+(k+2)!}$

o) $\sum_{k=1}^{n} \frac{4k}{4k^4+1}$

p) $\sum_{k=1}^{n} \frac{1}{(k+1)\sqrt{k}+k\sqrt{k+1}}$

q) $\sum_{k=1}^{n} k^2 C_n^k$

r) $\sum_{k=1}^{n} \frac{C_n^k}{k+1}$

s) $\sum_{k=1}^{n} \frac{C_n^k}{(k+1)(k+2)(k+3)}$

t) $\sum_{k=1}^{n} (-1)^k \frac{C_n^k}{(k+1)(k+2)}$

3. Bizonyítsa be a következő egyenlőtlenségeket!

a) $3n + 1 < 2log_2(n+2)$

b) $\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{n!} < \frac{5n-2}{2n}$

c) $\sqrt{n} < 2(\sqrt{n+1} - 1) < 1 + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}} < 2\sqrt{n} \quad n \geq 2$

d) $\frac{4^n}{n+1} < \frac{(2n)!}{(n!)^2} \quad n \geq 2$

e) $2^{n-1}(a^n + b^n) \geq (a+b)^n \quad a+b > 0; a; n \geq 1$

f) $(1 + \frac{1}{n})^k \geq 1 + \frac{k}{n}$

g) $(1 + \frac{1}{n})^k \geq 1 + \frac{k}{n} + \frac{k^2}{n^2} \quad k \leq n$

h) $2 \leq (1 + \frac{1}{n})^n$¡3

i) $\sqrt[n]{a_1\cdot a_2\cdot ...\cdot a_n} \leq \frac{a_1+a_2+...+a_n}{n} \quad a_i \in \mathbb{R}^+$

4. Bizonyítsa be a következő oszthatóságokat:

a) $30 \mid n^5 - n; \forall n \in \mathbb{N}$

b) $133 \mid 11^{n+2} + 12^{2n+1}; \forall n \in \mathbb{N}$

c) $9 \mid 4^n + 15n - 1; \forall n \in \mathbb{N}$

d) $17 \mid 3\cdot 5^{2n+1} + 2^{3n+1}; \forall n \in \mathbb{N}$

e) $31 \mid 6^{10n+1} + 15^{11n-1}; \forall n \in \mathbb{N}$